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REDUCING ABSTRACTION: THE CASE OF SCHOOL MATHEMATICS

ABSTRACT. There is a growing interest in the mathematics education community in the notion of abstraction and its significance in the learning of mathematics. "Reducing abstraction" is a theoretical framework that examines learners' behavior in terms of coping with abstraction level. It refers to situations in which learners are unable to manipulate concepts presented in a given problem; therefore, they unconsciously reduce the level of abstraction of the concepts involved to make these concepts mentally accessible. This framework has been used for explaining students' conception in different areas of undergraduate mathematics and computer science. This article extends the applicability scope of this framework from undergraduate mathematics to school mathematics. We draw on recently published research articles and exemplify how students' behavior can be described in terms of various interpretations of reducing abstraction level.

KEY WORDS: abstraction, cognitive theories, mathematics education, school mathematics

1. INTRODUCTION

The notion of abstraction in mathematics and in mathematical learning has recently received a lot of attention within the mathematics education research community. The significance of this topic, as well as the magnitude of community interest was highlighted at the Research Forum at the 2002 conference for the Psychology of Mathematics Education (PME) (Dreyfus and Gray, 2002). The purpose of that research forum was to discuss and critically compare three theories of abstraction, all aimed at providing a means for the description of the processes involved in the emergence of new mathematical mental structures. The forum was geared toward formulating an integrated theoretical framework that may serve to explain a vast collection of observations on mathematical thinking.

Building on the growing interest in the notion of abstraction, this article examines abstraction from the perspective of "reducing abstraction" – a mental process of coping with abstraction level of a given content or task. The theoretical framework of reducing abstraction (Hazzan, 1999) is usually associated with advanced mathematical thinking and topics in undergraduate mathematics. In this article it is utilized for the analysis of learners' understanding of school mathematics.

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We start with an overview of theories of abstraction in mathematical learning, aiming to describe the attention that this topic gets recently in the mathematics education research community. Further, we introduce the theme of reducing abstraction and situate it within the broad perspectives on mathematical abstraction. We then illustrate the application of this theme by examples taken from school mathematics. We conclude with the evaluation of reducing abstraction as a tool for analyzing mental activities of learners and with several suggestions for future research.

2. Theories of abstraction in mathematical learning

Abstraction is a complex concept that has many faces. As such, in a general context it has attracted the attention of many psychologists and educators (e.g., Beth and Piaget, 1966). In the more particular context of mathematics education research, abstraction has been discussed from a variety of viewpoints (cf. Tall, 1991; Noss and Hoyles, 1996; Frorer et al., 1997). There is no consensus with respect to a unique meaning for abstraction; however, there is an agreement that the notion of abstraction can be examined from different perspectives, that certain types of concepts are more abstract than others, and that the ability to abstract is an important skill for a meaningful engagement with mathematics.

The aforementioned research forum was assembled in an attempt to explore the variety of interpretations and the multi-faceted nature of abstraction. The theme of reducing abstraction, described in the next section, builds on this variety. Like other theories of abstraction, the theme of reducing abstraction focuses on learner's mental activities. We believe that it has "the potential to provide insight into one of the central aspects of learning mathematics and inform instructional practice" (Dreyfus and Gray, 2002, p. 113).

3. The theme of reducing abstraction

The theme of reducing abstraction (Hazzan, 1999) was originally developed to explain students' conception of abstract algebra. Abstract algebra is the first undergraduate mathematical course in which students "must go beyond learning 'imitative behavior patterns' for mimicking the solution of a large number of variations on a small number of themes (problems)" (Dubinsky et al., 1994, p. 268). Indeed, it is in the abstract algebra course that students are asked, for the first time, to deal with concepts that are introduced abstractly. That is, concepts are defined and presented by their properties and by an examination of "what facts can be determined just from (the properties) alone" (Dubinsky and Leron, 1994, p. 42). This new mathematical style of presentation requires learners to adopt new mental processes to cope with the new approach as well as with the new kind of mathematical objects. The theme of reducing abstraction emerged from an attempt to explain students' ways of thinking about abstract algebra concepts. The following description of the theme of reducing abstraction is largely based on Hazzan (1999).

The theme of reducing abstraction is based on three different interpretations for *levels of abstraction* discussed in the literature: (a) abstraction level as the quality of the relationships between the object of thought and the thinking person, (b) abstraction level as reflection of the process–object duality, and (c) abstraction level as the degree of complexity of the concept of thought. It is important to note that these interpretations of abstraction are neither mutually exclusive nor exhaustive. In what follows we briefly describe each of these three interpretations of abstraction.

(a) The interpretation of *abstraction level as the quality of the relationships between the object of thought and the thinking person* stems from Wilensky's (1991) assertion that whether something is abstract or concrete (or on the continuum between those two poles) is not an inherent property of the thing, "but rather a *property of a person's relationship to an object*" (p. 198). In other words, for each concept and for each person we may observe a different level of abstraction that reflects previous experiential connection between the two. The closer a person is to an object and the more connections he/she has formed to it, the more concrete (and the less abstract) he/she feels about it. On the basis of this perspective, some students' mental processes can be attributed to their tendency to make an unfamiliar idea more familiar or, in other words, to make the abstract more concrete.

This view is consistent with Hershkowitz, Schwarz and Dreyfus' (2001) perspective that emphasizes the learner's role in abstraction processes. They claim that "abstraction depends on the personal history of the solver" (p. 197). Specifically, based on Davydov's theory (1972/1990) they claim that "when a new structure is constructed, it already exists in a rudimentary form, and it develops through other structures that the learner has already constructed" (p. 219). Accordingly, abstraction is defined as "an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure." Vertical mathematization is "an activity in which mathematical elements are put together, structured, organized, developed, etc. into other elements, often in more abstract or formal form than the originals" (Hershkowitz et al., 1996 in Hershkowitz et al., 2001, p. 203).

(b) The interpretation of *abstraction level as reflection of the process–object duality* is based on the process–object duality, suggested by several theories of concept development in mathematics education (Beth and Piaget, 1966; Dubinsky, 1991; Sfard, 1991, 1992; Thompson, 1985). Some of these theories, such as the APOS (action, process, object and scheme) theory, suggest a more elaborate hierarchy (cf. Dubinsky, 1991). However, for our discussion it is sufficient to focus on the process–object duality. Theories that are based on this duality distinguish between a *process conception* and an *object conception* of mathematical notions, and, despite the differences, agree that when a mathematical concept is learned, its conception as a process precedes – and is less abstract than – its conception as an object (Sfard, 1991). Thus, process conception of a mathematical concept can be interpreted as being on a lower (i.e., reduced) level of abstraction than its conception as an object.

(c) The third interpretation of *abstraction level* examines abstraction *by the degree of complexity of the mathematical concept of thought*. For example, a set of elements is a more compound mathematical entity than any particular element in the set. It does not imply automatically, of course, that it should be more difficult to think in terms of compound objects. The working assumption here is that the more compound an entity is, the more abstract it is because a greater amount of detail has to be ignored when the entity is analyzed as a whole. In this respect, this interpretation of abstraction focuses on how students reduce abstraction level by replacing a set with one of its elements, thereby working with a less compound object.

The theme of reducing abstraction has been used so far for explaining students' conception in different areas of advanced mathematics and in computer science. Specifically, it was utilized to analyze learners' work in abstract algebra (Hazzan, 1999), differential equations (Raychaudhuri, 2001), data structures (Aharoni, 1999) and computability (Hazzan, 2003b). Hazzan (2003a) is a comprehensive report that illustrates the application of the theme of reducing abstraction in a variety of situations and topics taken from undergraduate mathematics and computer science. These works demonstrate that a wide range of cognitive phenomena can be explained by one theoretical framework. Here, we expand the applicability of the framework of reducing abstraction to the examination of students' learning of school mathematics.

4. Research method

As has been previously mentioned, our aim is to illustrate the applicability of the theme of reducing abstraction to students' learning of topics in school mathematics. Our data are drawn from two sources: (1) recent publications and (2) authors' personal experience.

(1) We have reviewed a variety of recently published articles in the mathematics education research literature, focusing on research that reports about students' attempts to cope with core topics from school mathematics. The selected data are taken from the following articles: Heirdsfield and Cooper, 2002; Kaminski, 2002; Karsenty, 2002; Knuth, 2002; Zazkis and Campbell, 1996; Zazkis et al., 2003. None of these research works used the framework of reducing abstraction, yet we found that the data they present can be naturally explained from the perspective of reducing abstraction. By applying this theoretical lens to the analysis of data that have already been analyzed using other theoretical perspectives, we aim to illustrate that the theme of reducing abstraction can enrich data interpretation. We believe that this further analysis illuminates additional aspects of learners' understanding of the discussed topics.

(2) For several years we have been engaged in teaching mathematics for preservice elementary school teachers, working toward teaching certificate. Naturally, students' work in these courses has become a major theme in our research. Though significant part of this research has been reported elsewhere and included in (1), we focus here on a few excerpts that have not been previously presented.

5. Reducing abstraction in school mathematics

We do not claim that any phenomenon can be explained from the perspective of reducing abstraction. However, we attempt to illustrate the applicability of the theme of reducing abstraction to a variety of topics from school mathematics. To do this, in this section we analyze a wide range of carefully selected examples (of the two kinds of data described in the previous section), by utilizing the framework of reducing abstraction, illuminating the three ways by which abstraction level may be reduced. In each case we specify the mathematical topic addressed and the source of the example. With respect to the excerpts of data taken from the mathematics education research literature, we first describe the research work presented in the original articles and the suggested analysis of the specific excerpts as it is presented in these articles. Then, we analyze the excerpt from the perspective of reducing abstraction.

Although the following examples are presented according to the three interpretations for levels of abstraction, this classification is somehow ambiguous. In other words, it is sometimes possible to describe a specific example of learners' behavior using different interpretations of reducing abstraction. We highlight this in Example 11 which is explained using three different perspectives on reducing the level of abstraction presented in this article. Other examples are presented according to a category that we perceive most appropriate.

(a) Quality of the relationships between the object of thought and the thinking person

Example 1: Linear functions (Karsenty, 2002)

Karsenty (2002) explores what adults remember from high school mathematics. She addresses this question focusing on linear functions. Responses to the task of drawing the graph of linear functions such as $y = 2x$, were documented and categorized. As it turns out, in many of these responses, the mathematical notion of linear graphing was replaced with personal on-the-spot constructing of ideas. The analysis presented in Karsenty's article is based on theories that explain the mechanism of recalling in terms of reconstruction versus reproduction.

We suggest that the data gathered in Karsenty's research are suitable for being analyzed through the lens of reducing abstraction in general and the interpretation of abstraction discussed in this section in particular. As it turns out, several of the adults participating in Karsenty's research could not make sense of the tasks presented to them; hence, all they could do was to rely on any familiar notions they found in the task, that is, to base their solution on an object with which they were familiar.

Here is one excerpt of data and its interpretation. Figure 1 presents Dov's attempts to draw the function $y = x$ (Karsenty, 2002, p. 127).

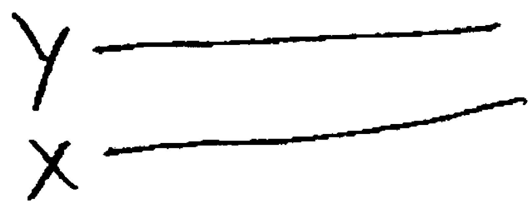

Figure 1. Dov's sketch for the function $y = x$ (Karsenty, 2002, p. 127).

Karsenty rightly categorizes this response as “describing the function through equality between shapes and line segments.” It is quite clear that Dov could not make sense of the task presented to him, and accordingly, all he could produce was a sketch that reflects some notion of equality. An examination from the perspective of reducing abstraction may reveal that this sketch is based on recalling that lines should appear in the graphing of expressions such as $y = x$. It is suggested that Dov could make sense of the notion of two parallel equal segments. Thus, relying on a familiar situation, Dov abandoned the unfamiliar object (the graph of $y = x$) and turned to face a familiar situation (two equal segments) of which he could make sense. Thus, the abstraction level is reduced.

Example 2: Different bases (Personal experience)
This example is taken from an interview with Sue, a preservice elementary school teacher who was introduced to the idea of calculation in bases other than ten.

Int: We’re in base five now. Can you add 12 and 14 (read: one-two and one-four) in base 5?
Sue: 12 (read: one-two) in base five is what? 7, yea, 5, 6, 7 and 14 (one-four) would be 9. So together this is 16.
Int: Is this in base 5?
Sue: Oh - no. I have to put this back into base 5. So 10 is 5, and we go 11, 12 (read: one-one, one-two, etc), 13, 14, 20... So I see, 20 is 10, and 30 will be 15 so 16 is 31, three-one base 5.

The reduction of the level of abstraction is illustrated by Sue’s tendency to retreat to the familiar base ten when asked to solve problems in terms of other bases. Since base ten is a familiar base with which students work through their entire school mathematics, this example refers to the relationships between the object of thought and the learner. Different bases are often used in courses for elementary school teachers to reinforce the common algorithms for multi-digit addition and subtraction and to create appreciation for the meaning of “carrying” and ”borrowing,” rather than just to perform these operations automatically following learned rules. However, as the above excerpt illustrates, Sue successfully avoids addition in base 5 by converting back to base 10, performing the operation in base 10 and then calculating the result in base 5. Her solution can be interpreted as an act of reducing abstraction from the unfamiliar base-5-addition to the familiar base-10-addition via conversion, which she achieved by counting and matching. From the perspective of abstraction presented in this section, abstraction level is reduced.

Example 3: Number sense (Kaminski, 2002)
Kaminski (2002) reports on pre-service primary teacher education students' involvement in a Number Sense program that was a component of a mathematics education unit. Kaminski's paper presents many observations focusing on class interaction. For example, the following quote refers to the specific phenomena of multiplication by zero:

"When discussing factors and exploring $76 \times 34 \times 0 \times 17$, Madie gained an insight into a long held misconception. Although 7×0 produces 0, the use of zero in the above four-term expression had been assumed by her for many years to indicate 'there was nothing to be multiplied by in the position occupied by zero.' Her solution was thus the result of multiplying $76 \times 34 \times 17$" (p. 137–138).

Kaminski explains that Madie's solution is caused by a misconception. Using the perspective of reducing abstraction we suggest two possible sources for this misconception.

First, it could be that Madie's interpretation for "the position occupied by zero" is influenced by her awareness of the positional system, that is, place value representation of numbers. Considering the expanded notation of the positional representation – as for example interpreting 7603 as $7 \times 1000 + 6 \times 100 + 0 \times 10 + 3 \times 1$ – the place value in the position of 0 indeed can be ignored, since "zero tens" means here "no tens."

Second, Madie's treatment of zero could be borrowed from addition exercises. When 0 is one of the addends, it indeed can be ignored. She is able to make sense of zeros in addition situations because addition situations are similar to real life situation: When something is empty, we ignore it. In other words, it is suggested that an explanation that relies on abstraction levels refers to Madie's need and inability to give some meaning to the appearance of 0 in multiplication exercises. As she could give such a meaning in situations of addition and place value interpretation, she leans on such situations and applies the same rule to multiplication.

Example 4: Conception of proofs (Knuth, 2002)
Knuth (2002) analyzes teachers' conceptions of proofs. Though a proof is a meta-mathematical concept, we include the analysis of its perception through the lens of reducing abstraction for at least two reasons: First, Knuth's research addresses teachers' conception of proofs. The fact that teachers' interpretation for the concept of proof can be interpreted as a reduction of the level of abstraction may have, in our opinion, direct implications in their teaching of proofs. Accordingly, we find it important to illuminate from an additional perspective teachers' conception of one of the central and basic mathematical tools. Second, a proof is by itself

an object which may be manipulated and worked with similarly to other mathematical objects. Thus, we suggest, it is not a significant difference whether a 'regular' mathematical object, such as a set or a function, or a 'meta-object' such as a proof, is discussed.

In what follows we present two teachers' descriptions of what a formal proof is, taken from Knuth's research. They illustrate how teachers reduce the level of abstraction by leaning on the pattern for proof, conventionally used in school, with which they are familiar from their own schooling.

Quote 1: "When I think of formal proof, I usually think of the two-column formal proof in geometry" (p. 71).

Quote 2: "When I think of a formal proof, I think of proofs where you have little 'T' (i.e., a spatial description for the organizational structure of a two column proof)" (p. 72).

Knuth (2002) explains the source of such definitions in the following way: "Also included in this group of nine teachers were those teachers (4) for whom two-column proofs (i.e., proofs in which statements are written in one column and the corresponding justifications in a second column) are the epitome of formal proofs" (p. 71).

The analysis through the lens of reducing abstraction adds to this explanation by highlighting the source for this conception. In the absence of mental construction for the concept of a formal proof, the teachers can only rely on their previous mathematical background, leaning on the main context in which they met proofs. In other words, such descriptions can be viewed as an expression of reducing abstraction by the *relationships between the object of thought and the thinking person*' interpretation, as these teachers conceive of the concept of proof as it is expressed in one limited context with which they are familiar, and this context represents for them the essence of proofs. Such a look may limit their ability to observe the essential characteristics of a proof.

(b) Process-object duality

Hazzan's (1999) contribution to the ongoing discussion about the process–object duality has highlighted two possible expressions of process conception of mathematical concepts, namely (a) students' personalization of formal expressions and logical arguments by using first-person language, and (b) students' tendency to work with canonical procedures in problem solving situations. Here, we illustrate the second expression with respect to school mathematics.

The term *canonical procedure* refers to a procedure that is more or less automatically triggered by a given problem. This can happen either because the procedure is naturally suggested by the nature of the problem, or because prior training has firmly linked a specific kind of problem with a specific procedure. The availability of a canonical procedure enables students to solve problems without analyzing properties of mathematical concepts, that is, to exhibit object conception, and to follow automatically the step-by-step algorithm that the canonical procedure provides.

One of the sources of students' tendency to use canonical algorithms is the intensive practice of solving the same kind of mathematical problems in class. As it turns out, this "step-by-step approach designed to yield mastery of the subject matter had the unfortunate consequence that students came to view themselves as the passive consumers of others' mathematics" (Schoenfeld, 1989, p. 341). Consequently, "students (feel) very strongly that mathematics always gives a rule to follow to solve problems" (Carpenter et al., 1983, pp. 656–657).

In what follows we present two examples for data analysis through the *process–object duality* interpretation of reducing abstraction. Though a wide variety of data is analyzed in the mathematics education research literature focusing on the process-object duality, our intention in presenting these examples is to illustrate how this analysis of data can be incorporated in a wider theoretical framework of analyzing learners' understanding of mathematical concepts.

Example 5: Elementary number theory (Zazkis and Campbell, 1996)
Zazkis and Campbell (1996) research preservice teachers' understanding of natural numbers, focusing on the properties of divisibility and multiplicative structure. In the following excerpt the preservice teacher works with the concept of divisibility.

Int: Consider the following number $3^3 \times 5^2 \times 7$. We'll talk about it a bit, so let's call it M. Is M divisible by 7, what do you think?
Mia: OK, I'll have to solve for M... [pause] Yes, it does.
Int: Would you please explain, what were you doing with your calculator?
Mia: I solved and this, this is 1575, and divided by 7 gives 225. Like it gives no decimal so 7 goes into it.

The tendency of students to calculate rather than attend to the structure of the number as represented in its prime decomposition has been discussed in detail in Zazkis and Campbell (1996). It has been reported that even students who are able to conclude divisibility of M by prime factors (in our case 3, 5 and 7) based on its structure, tend to calculation when prime non-factors (such as 11) or composite factors (such as 15 or 63) are in question. From the perspective of the process–object duality, these students reduce the level of abstraction by considering the process of divisibility, which is, attaining a whole number result in division, rather than analyzing the object of divisibility, which is a property of whole numbers that can be considered independently of any specific implementation of division.

Example 6: Addition and subtraction (Heirdsfield and Cooper, 2002)
Heirdsfield and Cooper report a study that focuses on two children's mental computation in addition and subtraction. Interviews were used to identify children's knowledge and ability with respect to number sense (including numeration, number and operations, basic facts, estimation), metacognition and affects. Both students were identified as being accurate. However, one student used a variety of mental strategies (was flexible) whereas the other student used only one strategy which reflected the written procedure for each of the addition and subtraction algorithms taught in the classroom.

Frameworks were developed to explain the two types of accuracy in mental addition and subtraction. Flexible accuracy was related to the presence of strong number sense knowledge integrated with metacognitive strategies and beliefs about self and teaching; whereas inflexible accuracy was a result of compensation of inadequate knowledge supported by beliefs about self and teaching.

The researchers focused on two students, Clare and Mandy. "The mental computation strategies used by Clare included separation (*left to right and right to left*) and *wholistic*. These strategies revealed numeration understanding, knowledge of the effect of operation on number, employment of estimation, and facility with number facts" (p. 61). "Mandy, like Clare was accurate in mental computation, but consistently employed *mental image of the pen and paper algorithm* (i.e., she imagined the numbers one under the other, as if using pen and paper). The individual numbers were first separated into place values and then operated on by moving right to left" (p. 65). The authors explain that "[i]t is posited that [. . .] Mandy used a procedure similar to the pen and paper algorithm, which required little understanding and with which she was very familiar. Further, Mandy's confidence in teacher–taught procedures resulted in her not seeing the need to develop understanding and more efficient strategies in other areas, as well. Thus, her beliefs also contributed to her satisfaction with the automatic traditional algorithm as her only strategy for mental computation" (p. 68).

We suggest that the above explanation fits very well into the framework of reducing abstraction in general and the process–object duality in particular. By applying these algorithms canonically, without referring to the properties of the objects under the examination, learners avoid treating the involved concepts as objects. That is, concepts are not thought about and manipulated through their properties (as an abstract approach would suggest), but rather, the abstraction level is reduced and problems are solved by using canonical algorithms.

(c) Degree of complexity of mathematical concepts

Example 7: Elementary number theory (Zazkis and Campbell, 1996)
The following excerpt is taken from the aforementioned study on preservice elementary school teachers' understanding of divisibility of natural numbers (Example 5).

Int: Do you think there is a number between 12358 and 12368 that is divisible by 7?
Nicole: I'll have to try them all, to divide them all, to make sure. Can I use my calculator?
Int: Yes, you may, but in a minute. Before you do the divisions, what is your guess?
Nicole: I really don't know. If it were 3 or 9, I could sum up the digits. But for 7 we didn't have anything like that. So I will have to divide them all.

Using the APOS (Action-Process-Object-Schema) theory (Dubinsky, 1991) the authors explained that Nicole had an 'action' conception of divisibility that is expressed by the need to carry out division explicitly to make a decision. At the next (process) stage the action can be imagined; however, with respect to divisibility, Nicole has not yet reached this stage.

From the perspective of reducing abstraction discussed in this section we suggest the following observation: Nicole wishes to find a number, divisible by 7, between the two given numbers, in order to claim its existence. While the task invites the consideration of an interval of 10 numbers, Nicole checks for divisibility of each number separately. In doing so she is considering particular numbers, rather than a more complex object, a set or interval of numbers. Therefore, the abstraction level is reduced: a property of a set is being examined by checking a property of each of its elements.

Example 8: Conception of proofs (Knuth, 2002)
Knuth's research (2002) about teachers' conceptions of proofs in the context of secondary school mathematics (see Example 4), observed also that

in lower level mathematics classes teachers accept informal proofs (i.e., empirically based arguments) as proof. The following teacher's comment is a representative explanation for this phenomenon: "When they say I noticed this pattern and I tested it out for quite a few cases; you tell them good job. For them, that's a proof. You don't bother them with these general cases" (Knuth, 2000, p. 76). Based on Harel and Sowder's (1998) analysis of empirical proof schemes, Knuth explains that "[a]n unfortunate consequence of such instruction, however, is that students may develop the belief that the verification of several examples constitutes proof" (p. 76).

We suggest that such a conception indicates a reduction in the complexity of the object of proof. This is derived from the fact that an argument is accepted as a proof of a specific theorem only when it is correct for *any* object that meets the conditions described in the theorem, not just for a specific set of objects for which the theorem is checked explicitly. From the perspective of reducing abstraction, learners' acceptance of a finite set of checks as a proof, is explained by their inability to construct, based on these specific checks, the object of proof, which is a more compound object that captures within it, among other cases, the specific instances that the learners checked explicitly.

This acceptance can be explained as means to cope with situations that require mental structures that have not yet been constructed in learners' mind. This statement is valid whether we talk about elementary, middle, high school or undergraduate mathematics. What may vary is the complexity and nature of objects for which learners are unable to construct mental structures. Although in more elementary mathematics this may be geometrical structure or number schemes, in more advanced mathematics these objects may be groups, differential equations or proofs. Still, the essence of the mental process is similar.

Example 9: Linear functions (Karsenty, 2002)

Example 1 addresses a response that Karsenty (2002) has classified in the category "Describing the function through equality between shapes and line segments." It provided an alternative analysis based on the interpretation of reducing abstraction by relying on familiar objects. Here we consider a response by another participant, Amira, which was classified by Karsenty in the category "Marking only one point in a coordinate system." In what follows we demonstrate how this example illustrates reduction of the level of abstraction by reducing the complexity of the object of though.

Amira is asked to draw the function y $= x$. "In response, she sketches a Cartesian axes system and marked the point $(1, 1)$" (p. 125). Here is Amira's reasoning: "You said that x is equal to y, and if this is x and this is y, and these are the positive points, and this is 1 and this is 1, so let's say I did it in the middle" (p. 125). Karsenty explains that this "response suggests that the request to draw the graph of $y = x$ is interpreted as "solving", i.e., finding a point in the Cartesian plane where y is indeed equal to x. The point marked is an arbitrary representative of solutions to the equation $y = x$" (p. 125).

We add to this interpretation, explaining it from the perspective of reducing abstraction described in this section. A function is a collection of ordered pairs that satisfy a given connection. This collection may be finite or infinite (pending on the function definition). In our case ($y = x$) the collection of ordered pairs is infinite. Accordingly, to grasp the essence of the graphic representation of the function, one has to construct mentally the object of an infinite set of ordered pairs.

In the present example, it seems that Amira remembered little from her school mathematics. Particularly, she could not follow any canonical algorithm to solve the problem she encountered. In such a case, when she lacked either any mental structure to lean on or any canonical procedure to follow, she *could* express this relation between x and y by pointing to one specific ordered pair. In other words, she replaced the infinite set of ordered pairs (which represents the function) with one ordered pair. As one ordered pair is clearly a less complex object than an infinite set of ordered pairs, according to the interpretation of abstraction described in this section, the level of abstraction is reduced.

Example 10: Translation of functions (Zazkis et al., 2003)
Zazkis et al., (2003) describe students' mental processes when they have to consider function transformation, that is, an operation that is applied to functions. The research focuses on horizontal translations and it describes participants' difficulty to explain the spatial location of (b) $y = (x - 3)^2$ in relation to (a) $y = x^2$. More specifically, participants acknowledge a conflict between the expectation that the graph of (b) should be located to the left of (a) and the knowledge (or realization, having checked the prediction) that it is located to the right of (a) when plotted on the same coordinate system.

Initially the authors use the notion of "obstacle" – either cognitive or epistemological – to explain participants' difficulty, identifying "tendency to generalize and possibly deceptive intuitions" as contributing constraints. Further, the authors strengthen their explanation by referring explicitly to the theme of reducing abstraction. They claim that the main source of difficulty here is in seeing the algebraic replacement (x moves to $x - 3$) as a transformation and trying to infer the geometric transformation, that is, the movement of the graph, from the algebraic substitution. That is to say that the transformation $f(x) \mapsto f(x - 3)$ is simplified to be considered as $x \mapsto (x - 3)$. In other words, students' attention is placed on the object of a variable (x) rather than on the more compound (hence, more abstract) object of a function ($f(x)$). As a function is a more compound object than a variable, this behavior can be considered as a reduction of the level of abstraction by the interpretation of abstraction presented in this section.

(d) Multifaceted examination from the perspective of reducing abstraction

Noss and Hoyles (1996) state that "[t]here is more than one kind of abstraction" (p. 49). Consequently, in this section we illustrate that there is more than one way to reduce the level of abstraction and more than one way to describe a learner's activity in terms of reducing abstraction level.

Example 11: Conversion of area units (personal experience)
As mentioned earlier, the classification of ways in which learner's reduce abstraction is neither exhaustive nor mutually exclusive. Consider for example the following problem:

> *A length of 3 cm on a scale model corresponds to a length of 10 m in a park. A lake in the park has an area of 3600 m^2. What is the area of the lake in the model?*

In her solution, Brenda assigned the dimensions 90×40 to the lake, converted each length separately and then calculated the area of lake in the model. Some of her classmates considered the lake to be a 36×100 rectangle or a 60×60 square. For most students, the random assignments of units and even the random restriction of the lake shape to either a square or a rectangle, still led to a correct answer. However, no one could explain why the final calculation of the area was not influenced by and specific choice of shape and measurements.

The task in this example was aimed at testing students' abilities to perform the conversion of square units. Regression to the units of length can be interpreted as reducing abstraction in several ways: In accordance with interpretation (c) of reducing abstraction, the assignment of units of lengths provides learners with a lesser degree of complexity of the object of thought. That is, it provides an opportunity to deal with one particular object rather than with *any* object of a given area. In accordance with interpretation (a) for reducing abstraction, the measures of lengths could have been perceived as more familiar, and therefore less abstract, than the measures of area. In accordance with interpretation (b) for reducing abstraction, the description of the area as a specific multiplication of two sizes, can be interpreted as students' conception of area as a process, rather than as an object that assigns a measure to a shape. In either case, students' regression to the units of length is a way of coping with the abstraction level presented by the task.

6. Conclusion

In this article we have enriched the ongoing discussion on the role of abstraction in learning mathematics by providing a different perspective on the notion, namely, reducing abstraction. Specifically, we have shown a wide range of strategies in dealing with presented tasks that can be interpreted as an act of "reducing the level of abstraction." The particular examples, taken from the core topics of school mathematics, ranged from early elementary addition and subtraction to upper secondary functions and transformations of functions.

Our contribution is two-fold: (a) we expand the applicability scope of abstraction theories in general and of reducing abstraction in particular by focusing on school mathematics, and (b) we extend the applicability of the theory by discussing mathematical meta-objects, such as the concept of proof. We suggest that the lens of reducing abstraction may be an applicable perspective for the analysis of the understanding of a wide collection of additional meta-objects, such as hypothesis, axiom, geometrical construction, etc. Future research will examine this applicability.

Schoenfeld (1998) proposed four major criteria for judging theories and models that embody them: descriptive power, explanatory power, predictive power and scope of applicability. We believe that the theory of reducing abstraction meets each of these criteria. First, it provides a lens for describing students' mental processes when facing problem–solving situations in which they are unable to cope at the expected level of abstraction. Second, the theory of reducing abstraction explains the mechanism by which students attempt or manage to cope with these situations, capitalizing on several interpretations of abstraction previously described in mathematics education literature. Third, the predictive power of the theory manifested itself in this article. That is, having previously applied the theory to topics in undergraduate mathematics and computer science, we predicted that it could illuminate students' behavior in dealing with mathematics at the school level and found a variety of examples to substantiate this prediction. Finally, the wide scope of applicability of this theory is expressed by the fact that it encapsulates under one umbrella a variety of phenomena. We extended this scope further by considering topics in school mathematics.

As stated earlier, the theme of reducing abstraction evolved from examining different interpretations for the level of abstraction discussed in the literature. However, the continuous evolution and refinement of the theories of abstraction in mathematics education may lead to further development and refinement of our framework.

We conclude by inviting the readers to examine their own observations of learners' mathematical encounters through the lens of reducing abstraction. This may include informal encounters with students' interpretations in learning or problem–solving situations or re-examination of previously analyzed data though a different perspective.

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